

# Supersymmetric solutions of 4-dimensional supergravities

Tomás Ortín <sup>1</sup>

*Instituto de Física Teórica UAM/CSIC  
Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049 Madrid, Spain*

## Abstract

We review general and recent results on the characterization and construction of timelike supersymmetric solutions of 4-dimensional supergravity theories.

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<sup>1</sup>E-mail: Tomas.Ortin@cern.ch

# 1 Introduction: rôle and importance of supersymmetric solutions

Supersymmetry arose almost 40 years ago as a possible symmetry of Nature that would unify the two seemingly different types of elementary constituents of the Universe: matter and interactions (*i.e.* bosons  $\phi^b$  and fermions  $\phi^f$ ). Much of the interest in supersymmetry is due to the fact that its presence is crucial for the consistency of (super-) string theories (and gravity/gauge (AdS/CFT) correspondences) on different vacua.

The local (*gauge*) generalization of supersymmetry, aptly named *supergravity* requires/implies the coupling of all the fields to standard (General Relativity (GR)) gravity which is, then, included in the unification. From a more pedestrian point of view supergravity theories can be seen as nothing but extensions of GR consisting in a number of fermionic and bosonic fields coupled to gravity. Many purely bosonic theories can be supersymmetrized (or *embedded* in a supersymmetric theory) by the simple addition of fermionic fields in appropriate numbers and species and with appropriate couplings. The cosmological Einstein-Maxwell theory, for instance, can be embedded in  $N = 1, d = 4$  supergravity in a number of ways [1].

At low energies, superstring theories can be effectively described by supergravity theories<sup>2</sup>. This leads to an extremely rich interplay between superstring and supergravity theories which has allowed, for instance, to prove the UV finiteness of  $N = 8$  supergravity to the fourth loop order, raising the possibility that it may be a finite quantum field theory of gravity [2].

In this talk we are interested in purely bosonic ( $\phi^f = 0$ ) solutions of the classical equations of motion of supergravity theories. Since  $\phi^f = 0$  is always a consistent truncation, the solutions of the truncated supergravity theory (GR coupled to some bosonic fields) are automatically solutions of the full supergravity theory. Thus, for instance, all the standard solutions of the cosmological Einstein-Maxwell theory are also purely bosonic solutions of  $N = 1, d = 4$  supergravity and *vice versa*. These solutions are also important from the superstring theory point of view: the theory can only be quantized consistently in backgrounds (*vacua*) which are solutions or their associated supergravity description<sup>3</sup>. Furthermore, the *supersymmetric* solutions, to be defined later, can also be interpreted as the long-range fields generated by a source which is a state of the superstring theory. The identification of the sources of *supersymmetric* (a.k.a. BPS) black holes in terms of states of superstring theory on a suitable background is the keystone of the microscopic interpretation (via the “gauge dual”) of these black hole’s entropy.

The (unbroken) supersymmetry of the classical solution plays a crucial role in this and many other problems. This is what makes supersymmetric (BPS) solutions interesting. Many interesting and well-known GR solutions (Minkowski, (Anti-) De Sitter, extreme Reissner-Nordström, *pp*-waves...) are supersymmetric. Let us define this property: a bosonic field configuration of a supergravity theory (no necessarily solving its equations of motion) is *supersymmetric* if it is invariant under some supersymmetry transformations. The transformations generated by the

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<sup>2</sup>The reverse is not always true.

<sup>3</sup>We would like to stress that the proliferation of possible string theory vacua, the so-called *landscape problem*, is actually common to all theories containing GR.

spinor  $\epsilon^\alpha(x)$  take the generic form

$$\delta_\epsilon \phi^b \sim \bar{\epsilon} \phi^f, \quad \delta_\epsilon \phi^f \sim \partial \epsilon + (\phi^b + \bar{\phi}^f \phi^f) \epsilon, \quad (1.1)$$

Then, a bosonic configuration is invariant under the transformation generated by  $\epsilon^\alpha(x)$  if it satisfies the *Killing Spinor Equations (KSEs)*

$$\delta_\epsilon \phi^f \sim \partial \epsilon + \phi^b \epsilon = 0. \quad (1.2)$$

This is a generalization of the concept of isometry, an infinitesimal g.c.t. generated by  $\xi^\mu(x)$  that leaves the metric  $g_{\mu\nu}$  invariant because it satisfies the Killing Vector Equation. Each isometry is associated to a bosonic generator of a (Lie) symmetry algebra

$$\xi_{(I)}^\mu(x) \rightarrow P_I, \quad [P_I, P_J] = f_{IJ}{}^K P_K. \quad (1.3)$$

Correspondingly, each supersymmetry is associated to an odd generator of a (Lie) symmetry superalgebra

$$\epsilon_{(n)}^\alpha(x) \rightarrow Q_n, \quad [Q_n, P_I] = f_{nI}{}^m Q_m, \quad \{Q_n, Q_m\} = f_{nm}{}^I P_I. \quad (1.4)$$

Every supersymmetric field configuration has a supersymmetry superalgebra. For instance, the superalgebra of Minkowski spacetime is the Poincaré superalgebra with

$$\{Q_\alpha, Q_\beta\} = (\gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu. \quad (1.5)$$

The supersymmetric solutions have a number of interesting properties:

1. They saturate BPS bounds like  $M = |Q|$  (extreme Reissner-Nordström solution).
2. Multicenter supersymmetric solutions are possible (Majumdar-Papapetrou multi-R-N-black hole solution) thanks to the equilibrium of forces  $M_i M_j = Q_i Q_j$ .
3. Their sources (possibly, branes) can be identified.
4. They enjoy classical and quantum stability: results can be extrapolated to different domains (invariance under *dualities*).
5. They are more symmetric and have simpler functional forms that depend on a smaller number of independent functions.
6. They are easier to find: the off-shell equations of motion of supersymmetric configurations are related by the *Killing Spinor Identities (KSIs)* [3]: if we denote the (l.h.s. of the) bosonic equations of motion by  $\mathcal{E}(\phi^b) \equiv \left. \frac{\delta S}{\delta \phi^b} \right|_{\phi^f=0}$  for a supersymmetric field configuration with Killing spinor  $\epsilon$ ,  $\delta_\epsilon \phi^f|_{\phi^f=0}$  they are

$$\mathcal{E}(\phi^b) (\delta_\epsilon \phi^b)_{,f1}|_{\phi^f=0} = 0. \quad (1.6)$$

These relations between the off-shell bosonic equations of motion  $\mathcal{E}(\phi^b)$  are necessary conditions for supersymmetry. We only need to check a few equations of motion on a supersymmetric configuration. The KSIs also constrain the possible sources enforcing cosmic censorship if we require them to hold everywhere in spacetime [4]. Finally, they provide powerful consistency checks when we try to find large families of supersymmetric solutions, as we are going to do.

7. In supersymmetric black-hole solutions there is an *attractor mechanism* which suppresses *primary scalar hair* and hints at a microscopic interpretation of the entropy [5]: consider a supersymmetric, static, spherically symmetric, asymptotically flat, black-hole solution given by the fields

$$\{g_{rr}(r), F_{tr}^\Lambda(r), \star F_{tr}^\Lambda(r), \phi^i(r)\}. \quad (1.7)$$

These solutions are fully characterized by the electric and magnetic charges  $q_\Lambda, p^\Lambda$  and the asymptotic values of the scalars  $\phi_\infty^i$ . Supersymmetry imposes the saturation of the BPS bound:  $M = f(q_\Lambda, p^\Lambda, \phi_\infty^i)$  for some function  $f$ . It can be shown that *at the event horizon*  $r = r_H$  the scalars  $\phi^i$  and the metric function  $r^2 g_{rr}$  take their *attractor* value which depends on the conserved charges  $q_\Lambda, p^\Lambda$  and not on  $\phi_\infty^i$ :

$$\phi^i(r_H) = \phi_{\text{attract}}^i(q, p), \quad r_H^2 g_{rr}(r_H) = 4\pi S(q, p). \quad (1.8)$$

This proves that, at least for these supersymmetric black holes, the Bekenstein-Hawking entropy  $S(q, p)$  only depends on charges which will be quantized, and therefore it is just a function of integers amenable to a microscopic interpretation.

## 2 The search for all 4-d supersymmetric solutions

In his pioneering work [6] Tod showed that it is possible find all the BPS solutions of pure  $N = 2, d = 4$  supergravity (i.e. solutions of the Einstein-Maxwell theory). His result uses the doublet of spinors (we are in  $N = 2$ ) as a basis in the Newman-Penrose formalism and has been generalized (using the spinor-bilinear method developed in Ref. [7]<sup>4</sup>) to include the coupling of more scalar and vector fields [9], cosmological constant [10] and non-Abelian symmetries [11]. The supersymmetric solutions found include regular black holes with non-Abelian fields, not in numerical form as those in Refs. [12], but in completely analytic form.

For  $N > 2$  there are further spinors containing information that cannot be extracted with the Newman-Penrose formalism. In [8] it was shown how to overcome those problems and determine the form of all the timelike supersymmetric solutions<sup>5</sup> of all  $d = 4$  ungauged supergravities using the spinor-bilinear method. Since they turn out to be related to those of  $N = 2$  theories [11], we briefly review them first.

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<sup>4</sup>See the references in [8] for other methods and results in dimensions other than 4.

<sup>5</sup>The  $N \geq 2$  supersymmetric solutions fall in two cases: *null* and *timelike*.

## 2.1 The N=2 case

The  $N = 2$  *supergravity multiplet* is

$$\{e^a_\mu, \psi_{I\mu}, A^{IJ}_\mu\}, \quad I, J, \dots = 1, 2, \Rightarrow A^{IJ}_\mu = A^0_\mu \varepsilon^{IJ}. \quad (2.1)$$

We can couple to it  $n$  *vector multiplets*

$$\{A^i_\mu, \lambda^i_I, Z^i\}, \quad i = 1, \dots, n, \Rightarrow A^\Lambda_\mu, \quad \Lambda = 0, \dots, n,$$

where the  $Z^i$ s are complex scalars, and  $m$  *hypermultiplets*

$$\{\zeta_\alpha, q^u\}, \quad u = 1, \dots, 4m, \quad \alpha = 1, \dots, 2m. \quad (2.2)$$

The  $n$   $Z^i$ s are encoded into the  $2\bar{n}$ -dimensional symplectic section ( $\bar{n} = 1 + n$ )

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}, \quad \langle \mathcal{V} | \mathcal{V}^* \rangle = -2i. \quad (2.3)$$

This description of the scalars is extremely redundant but useful.

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|} \left[ R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*j^*} + 2H_{uv} \partial_\mu q^u \partial^\mu q^v + 2\Im \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\mu\nu} F^\Sigma_{\mu\nu} - 2\Re \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\mu\nu} \star F^\Sigma_{\mu\nu} \right], \quad (2.4)$$

where  $\mathcal{G}_{ij^*}$  is the Kähler metric parametrized by the  $Z^i$ s etc.<sup>6</sup>

The  $N = 2, d = 4$  KSEs take the form

$$\begin{aligned} \delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ} T^+_{\mu\nu} \gamma^\nu \epsilon^J = 0, \\ \delta_\epsilon \lambda^{iI} &= i \not{\partial} Z^i \epsilon^I + \varepsilon^{IJ} \mathcal{Q}^{i+}_J \epsilon_J = 0, \\ \delta_\epsilon \zeta_\alpha &= -i \mathbb{C}_{\alpha\beta} U^{\beta I}_u \varepsilon_{IJ} \not{\partial} q^u \epsilon^J = 0, \end{aligned} \quad (2.5)$$

where  $(\langle \cdot | \cdot \rangle)$  is the symplectic product)

$$T^+ = \langle \mathcal{V} | \mathcal{F}^+ \rangle, \quad G^{i+} = \frac{i}{2} \mathcal{G}^{ij^*} \langle \mathcal{D}_{j^*} \mathcal{V}^* | \mathcal{F}^+ \rangle, \quad \mathcal{F}^+ \equiv \begin{pmatrix} F^{\Lambda+} \\ \mathcal{N}_{\Lambda\Sigma}^* F^{\Sigma+} \end{pmatrix}, \quad (2.6)$$

and  $\mathfrak{D}$  is the Lorentz-Kähler- $SU(2)$ -covariant derivative ( $U(1)_{\text{Kähler}} + SU(2) = U(2)$ )

$$\mathfrak{D}_\mu \epsilon_I = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{2} \mathcal{Q}_\mu \right) \epsilon_I + A_{\mu I}^J \epsilon_J, \quad (2.7)$$

and where  $U^{\alpha I}_u(q)$  is the quaternionic-Kähler  $4m$ -*bein* [13].

Our goal is to find all the bosonic field configurations  $\{e^a_\mu, A^\Lambda_\mu, Z^i, q^u\}$  such that the above KSEs admit at least one solution  $\epsilon^I(x)$ . In the *spinor-bilinear method* [7] we

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<sup>6</sup>See Ref. [13] for a complete description of all the objects that appear in this action.

1. Assume that one has a bosonic field configuration such that one solution  $\epsilon^I$  exists.
2. Construct all the independent spinor bilinears with the commuting  $\epsilon^I$  and find the equations they satisfy: (a) Due to the Fierz identities. (*spinor-bilinear algebra*) and (b) Due to the KSEs.
3. Find their integrability conditions and show that they are also sufficient to solve the KSEs. At this point all supersymmetric configurations are determined.
4. Determine with the KSIs which equations of motion are independent for supersymmetric configurations and impose them on the supersymmetric configurations we just identified.

The independent bilinears that we can construct with the  $U(2)$  doublet of Weyl spinors  $\epsilon_I$   $I = 1, 2$  are:

1. A complex antisymmetric matrix of scalars  $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$ .  $X$  is an  $SU(2)$  singlet but has  $U(1)$  Kähler weight.
2. A Hermitean matrix of vectors  $V^I_{Ja} \equiv i \bar{\epsilon}^I \gamma_a \epsilon_J$ .

The 4-d Fierz identities imply that  $V_a \equiv V^I_{Ia}$  is always non-spacelike:

$$V^2 = -V^I_J \cdot V^J_I = 2M^{IJ}M_{IJ} = 4|X|^2 \geq 0. \quad (2.8)$$

We only consider the timelike case  $X \neq 0$  in which all  $V^I_{Ja}$  are independent. With them,  $\sigma^0 \equiv 1$  and the Pauli matrices  $\sigma^m$  one can construct an orthonormal tetrad

$$V^a_\mu \equiv \frac{1}{\sqrt{2}} V^I_{J\mu} (\sigma^a)^J_I, \quad V^I_{J\mu} = \frac{1}{\sqrt{2}} V^a_\mu (\sigma^a)^I_J, \quad (2.9)$$

in which  $V^0 = \sqrt{2}V$  is timelike and the  $V^m$ s are spacelike.

Observe that this construction does not work for  $N > 2$  where we have  $U(N)$  vectors of spinors and we can only select 2 of them at the expense of breaking manifest  $U(N)$  invariance.

Apart from these equations (part of the spinor-bilinear algebra) the bilinears  $X$  and  $V^I_J$  satisfy a number of equations that follow from the assumption that  $\epsilon^I$  satisfies the KSEs (is a Killing spinor). They can be found in Refs. [11].

If we denote the (l.h.s. of the) Einstein, Maxwell (and Bianchi) and scalar equations of motion by  $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^\mu, \mathcal{E}^i, \mathcal{E}_u\}$  resp., then the KSIs of this theory imply:

1.  $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0$ .
2.  $\mathcal{E}^m = 0$ .
3.  $\mathcal{E}_u = 0$ . This implies absence of attractor mechanism for the hyperscalars  $q^u$ .
4.  $\mathcal{E}^{00} = -4|X| \langle \mathcal{E}^0 | \Re(\mathcal{V}/X) \rangle$ . This is related to the BPS bound.
5.  $0 = \langle \mathcal{E}^0 | \Im(\mathcal{V}/X) \rangle$ . This implies the absence of sources of NUT charge [4].

6.  $\mathcal{E}_{i^*} = 2 \left( \frac{X}{X^*} \right)^{1/2} \langle \mathcal{E}^0 \mid \mathcal{D}_{i^*} \mathcal{V}^* \rangle$ . This implies the existence of an attractor mechanism for the complex scalars  $Z^i$ .

The only independent equations of motion that have to be imposed on  $N = 2, d = 4$  supersymmetric configurations are, therefore, the zeroth components of the Maxwell equations and Bianchi identities:  $\mathcal{E}^0 = 0$ .

The result can be summarized in the following recipe: all the supersymmetric solutions of a  $N = 2, d = 4$  supergravity can be constructed as follows:

1. Define the  $U(1)$ -neutral real symplectic vectors  $\mathcal{R}$  and  $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X. \quad (2.10)$$

No Kähler nor  $SU(2)$  gauge-fixing are necessary.

2. The components of  $\mathcal{I}$  are given by a symplectic vector real functions  $\mathcal{H}$  harmonic in the 3-dimensional transverse space with metric  $\gamma_{\underline{mn}}$ :

$$\nabla_{(3)}^2 \mathcal{H} = 0. \quad (2.11)$$

3.  $\mathcal{R}$  can be found from  $\mathcal{I}$  by using the redundancy of the description provided by  $\mathcal{V}$  which implies the existence of relations between  $\mathcal{R}$ s and  $\mathcal{I}$ s known as *stabilization equations* and which may be very difficult to solve in practice.

4. The scalars  $Z^i$  are given by the quotients

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}. \quad (2.12)$$

5. The hyperscalars  $q^u(x)$  are given by any mapping satisfying

$$\mathcal{U}^{\alpha J}_m (\sigma^m)_{J^I} = 0, \quad \mathcal{U}^{\alpha J}_n \equiv V_n^{\underline{m}} \partial_{\underline{m}} q^u \mathcal{U}^{\alpha J}_u. \quad (2.13)$$

6. The metric takes the form

$$ds^2 = 2|X|^2(dt + \omega)^2 - \frac{1}{2|X|^2} \gamma_{\underline{mn}} dx^m dx^n. \quad (2.14)$$

where

$$|X|^{-2} = 2 \langle \mathcal{R} \mid \mathcal{I} \rangle, \quad (d\omega)_{mn} = 2\epsilon_{mnp} \langle \mathcal{I} \mid \partial^p \mathcal{I} \rangle. \quad (2.15)$$

$\gamma_{\underline{mn}}$  is determined indirectly from the hyperscalars: its spin connection  $\varpi^{mn}$  in the basis  $\{V^m\}$  is related to the pullback of the  $SU(2)$  connection of the hyper-Kähler manifold  $A^I_{J\mu} = \frac{1}{\sqrt{2}} A^m_u (\sigma^m)^I_J \partial_\mu q^u$ , by

$$\varpi_m^{np} = \epsilon^{npq} A^q_m. \quad (2.16)$$

7. The vector field strengths are

$$\mathcal{F} = -\frac{1}{2} d(\mathcal{R}\hat{V}) - \frac{1}{2} \star (\hat{V} \wedge d\mathcal{I}), \quad \hat{V} = 2\sqrt{2}|X|^2(dt + \omega). \quad (2.17)$$

### 3 The all- $N$ case

Dealing with all the  $N > 1, d = 4$  supergravities simultaneously is possible thanks to the formalism developed in Ref. [14] which generalizes that of the  $N = 2, d = 4$  theories. It turns out that all 4-d supergravity multiplets can be written in the form

$$\{e^a{}_\mu, \psi_{I\mu}, A^{IJ}{}_\mu, \chi_{IJK}, P_{IJKL\mu}, \chi^{IJKLM}\}, \quad I, J, \dots = 1, \dots, N, \quad (3.1)$$

and all vector multiplets can be written in the form

$$\{A_{i\mu}, \lambda_{iI}, P_{iIJ\mu}, \lambda_i^{IJK}\}, \quad i = 1, \dots, n, \quad (3.2)$$

where  $P_{IJKL\mu}, P_{iIJ\mu}$  are the pullbacks of the scalar Vielbeins. The price to pay for using this representation is that there is some redundancy: all the fields that can be related by  $SU(N)$  duality relations, are:

- $N = 4$ :  $P^{*IJ} = \frac{1}{2}\varepsilon^{IJKL}P_{iKL}$ , and  $\lambda_{iI} = \frac{1}{3!}\varepsilon_{IJKL}\lambda_i^{IJK}$ .
- $N = 6$ :  $P^{*IJ} = \frac{1}{4!}\varepsilon^{IJK_1\dots K_4}P_{K_1\dots K_4}$ ,  $\chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{IJK}$ ,  $\chi^{I_1\dots I_5} = \varepsilon^{I_1\dots I_5J}\lambda_J$ .
- $N = 8$ :  $P^{*I_1\dots I_4} = \frac{1}{4!}\varepsilon^{I_1\dots I_4J_1\dots J_4}P_{J_1\dots J_4}$ ,  $\chi_{I_1I_2I_3} = \frac{1}{5!}\varepsilon_{I_1I_2I_3J_1\dots J_5}\chi^{J_1\dots J_5}$ .

All these constraints must be carefully taken into account.

The scalars are encoded into the  $2\bar{n}$ -dimensional ( $\bar{n} \equiv n + \frac{N(N-1)}{2}$ ) symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f_{IJ}^\Lambda \\ h_{\Lambda IJ} \end{pmatrix}, \quad \text{and} \quad \mathcal{V}_i = \begin{pmatrix} f_i^\Lambda \\ h_{\Lambda i} \end{pmatrix}, \quad \Lambda = 1, \dots, \bar{n}, \quad (3.3)$$

normalized

$$\langle \mathcal{V}_{IJ} | \mathcal{V}^{*KL} \rangle = -2i\delta^{KL}{}_{IJ}, \quad \langle \mathcal{V}_i | \mathcal{V}^{*j} \rangle = -i\delta_i^j. \quad (3.4)$$

They can be combined into the  $Usp(\bar{n}, \bar{n})$  matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^* + ih^* \\ f - ih & f^* - ih^* \end{pmatrix}. \quad (3.5)$$

They reduce to the standard sections of the  $N = 2$  case

$$\mathcal{V}_{IJ} = \mathcal{V}\varepsilon_{IJ}, \quad = \begin{pmatrix} \mathcal{L}^\Lambda \varepsilon_{IJ} \\ \mathcal{M}_\Lambda \varepsilon_{IJ} \end{pmatrix}, \quad \text{and} \quad \mathcal{V}_i = \mathcal{D}_i \mathcal{V} = \begin{pmatrix} f_i^\Lambda \\ h_{\Lambda i} \end{pmatrix}. \quad (3.6)$$

The graviphotons  $A^{IJ}{}_\mu$  do not appear directly, only through the “dressed” vectors

$$A^\Lambda{}_\mu \equiv \frac{1}{2}f^\Lambda{}_{IJ}A^{IJ}{}_\mu + f^\Lambda{}_i A^i{}_\mu. \quad (3.7)$$

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|} \left[ R + 2\Im \mathcal{N}_{\Lambda\Sigma} F^\Lambda{}^{\mu\nu} F^\Sigma{}_{\mu\nu} - 2\Re \mathcal{N}_{\Lambda\Sigma} F^\Lambda{}^{\mu\nu} \star F^\Sigma{}_{\mu\nu} + \frac{2}{4!}\alpha_1 P^{*IJKL}{}_\mu P_{IJKL}{}^\mu + \alpha_2 P^{*IJ}{}_\mu P_{IJ}{}^\mu \right], \quad (3.8)$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T, \quad h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma, \quad \mathfrak{D}h_\Lambda = \mathcal{N}_{\Lambda\Sigma}^* \mathfrak{D}f^\Lambda. \quad (3.9)$$

The  $N$ -specific constraints must be taken into account to find the e.o.m.:



**For  $N = 2$ :**  $\mathcal{E}^{IJ} = \mathfrak{D}^\mu P^{*IJ}{}_\mu + 2T^{i-}{}_{\mu\nu} T^{IJ-\mu\nu} + P^{*IJ}{}^A P^{*jk}{}_A T_j^+{}_{\mu\nu} T_k^{+\mu\nu}$ .

**For  $N = 3$ :**  $\mathcal{E}^{IJ} = \mathfrak{D}^\mu P^{*IJ}{}_\mu + 2T^{i-}{}_{\mu\nu} T^{IJ-\mu\nu}$ .

$$\text{For } N = 4: \begin{cases} \mathcal{E}^{IJKL} = \mathfrak{D}^\mu P^{*IJKL}{}_\mu + 6T^{[IJ] -}{}_{\mu\nu} T^{[KL] -\mu\nu} \\ \quad + P^{*IJKLA} P^{*ij}{}_A T_i^+{}_{\mu\nu} T_j^{+\mu\nu}, \\ \mathcal{E}^{IJ} = \mathfrak{D}^\mu P^{*IJ}{}_\mu + T^{i-}{}_{\mu\nu} T^{IJ-\mu\nu} + \frac{1}{2}\varepsilon^{IJKL} T_i^+{}_{\mu\nu} T_{KL}^{+\mu\nu}, \end{cases}$$

etc.

The supersymmetry transformations of the fermionic fields are

$$\begin{aligned} \delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + T_{IJ}^+{}_{\mu\nu} \gamma^\nu \epsilon^J, \\ \delta_\epsilon \chi_{IJK} &= -\frac{3i}{2} \mathcal{T}_{[IJ}^+ \epsilon_{K]} + i \mathcal{P}_{IJKL} \epsilon^L, \\ \delta_\epsilon \lambda_{iI} &= -\frac{i}{2} \mathcal{T}_i^+ \epsilon_I + i \mathcal{P}_{iIJ} \epsilon^J, \\ \delta_\epsilon \chi_{IJKLM} &= -5i \mathcal{P}_{[IJKL} \epsilon_{M]} + \frac{i}{2} \varepsilon_{IJKLMN} \mathcal{T}^- \epsilon^N + \frac{i}{4} \varepsilon_{IJKLMNOP} \mathcal{T}^{NO-} \epsilon^P, \\ \delta_\epsilon \lambda_{iIJK} &= -3i \mathcal{P}_{i[IJ} \epsilon_{K]} + \frac{i}{2} \varepsilon_{IJKL} \mathcal{T}_i^- \epsilon^L + \frac{i}{4} \varepsilon_{IJKLMN} \mathcal{T}^{LM-} \epsilon_N, \end{aligned} \tag{3.10}$$

where the graviphoton and matter vector field strengths are

$$T_{IJ}^+ = \langle \mathcal{V}_{IJ} | \mathcal{F}^+ \rangle, \quad T_i^+ = \langle \mathcal{V}_i | \mathcal{F}^+ \rangle, \quad \mathcal{F}_\Lambda^+ = \mathcal{N}_{\Lambda\Sigma}^* F^{\Sigma+}, \tag{3.11}$$

and where

$$\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu^J{}_I, \tag{3.12}$$

and  $\Omega_\mu^J{}_I$  is the pullback of the connection of the scalar manifold ( $\subset U(N)$ ).

For all values of  $N$  the independent KSEs take the form

$$\begin{aligned} \mathfrak{D}_\mu \epsilon_I + T_{IJ}^+{}_{\mu\nu} \gamma^\nu \epsilon^J &= 0, \\ \mathcal{P}_{IJKL} \epsilon^L - \frac{3}{2} \mathcal{T}_{[IJ}^+ \epsilon_{K]} &= 0, \\ \mathcal{P}_{iIJ} \epsilon^J - \frac{1}{2} \mathcal{T}_i^+ \epsilon_I &= 0, \\ \mathcal{P}_{[IJKL} \epsilon_{M]} &= 0, \\ \mathcal{P}_{i[IJ} \epsilon_{K]} &= 0. \end{aligned} \tag{3.13}$$

The last two KSEs should only be considered for  $N = 5$  and  $N = 3$ , resp.

Again, our goal is to find all the bosonic field configurations  $\{e^a{}_\mu, A^\Lambda{}_\mu, P_{IJKL\mu}, P_{iIJ\mu}\}$  such that the above KSEs admit at least one solution  $\epsilon^I$  following the spinor-bilinear method. We only consider the timelike case.

First, we construct all the possible independent bilinears with one  $U(N)$  vector of Weyl spinors  $\epsilon_I$  and compute the spinor-bilinears algebra. The independent bilinears are:

1. A complex antisymmetric matrix of scalars  $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$ .
2. A Hermitean matrix of vectors  $V^I{}_{Ja} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$ .

The Fierz identities imply the following properties for them:

1.  $M_{I[J}M_{KL]} = 0$ , so  $\text{rank}(M_{IJ}) \leq 2$ .
2.  $V_a \equiv V^I_{Ia}$  is always non-spacelike:  $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \geq 0$ .
3. We can choose a tetrad  $\{e^a_\mu\}$  such that  $e^0_\mu \equiv \frac{1}{\sqrt{2}}|M|^{-1}V_\mu$ . Then, *defining*  $V^m_\mu \equiv |M|e^m_\mu$  we can decompose

$$V^I_{J\mu} = \frac{1}{2}\mathcal{J}^I_J V_\mu + \frac{1}{\sqrt{2}}(\sigma^m)^I_J V^m_\mu, \quad (3.14)$$

where  $\mathcal{J}^I_J = 2M^{IK}M_{JK}|M|^{-2}$  is a rank 2 projector (Tod):

$$\mathcal{J}^2 = \mathcal{J}, \quad \mathcal{J}^I_I = +2, \quad \mathcal{J}^I_J \epsilon^J = \epsilon^I. \quad (3.15)$$

The main properties satisfied by the three  $\sigma^m$  matrices are:

$$\begin{aligned} \sigma^m \sigma^n &= \delta^{mn} \mathcal{J} + i\varepsilon^{mnp} \sigma^p, \\ \mathcal{J} \sigma^m &= \sigma^m \mathcal{J} = \sigma^m, \\ (\sigma^m)^I_I &= 0, \\ \mathcal{J}^K_J \mathcal{J}^L_I &= \frac{1}{2} \mathcal{J}^K_I \mathcal{J}^L_J + \frac{1}{2} (\sigma^m)^K_I (\sigma^m)^L_J, \\ M_{K[I} (\sigma^m)^K_{J]} &= 0, \\ 2|M|^{-2} M_{LI} (\sigma^m)^I_J M^{JK} &= (\sigma^m)^K_L, \end{aligned} \quad (3.16)$$

Summarizing:  $\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$  is an  $x$ -dependent basis of a  $\mathfrak{u}(2)$  subalgebra of  $\mathfrak{u}(N)$  in the 2-dimensional eigenspace of  $\mathcal{J}$  of eigenvalue  $+1$  and provide a basis in the space of Hermitean matrices  $A$  satisfying  $\mathcal{J}A\mathcal{J} = A$

These bilinears also satisfy a number of equations that follow from the assumption that  $\epsilon^I$  is a Killing spinor. They can be found in Ref. [8]. Also from this assumption follow the KSIs that constrain the (l.h.s. of the) Einstein equations, Maxwell equations and Bianchi identities and scalar equations of motion, denoted by  $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^\mu, \mathcal{E}^{IJKL}, \mathcal{E}^{iIJ}\}$  resp. Defining  $\tilde{\mathcal{J}}^I_J \equiv \delta^I_J - \mathcal{J}^I_J$  we get

$$1. \mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$$

$$2. \mathcal{E}^m = 0.$$

$$3. \begin{cases} \mathcal{E}^{MNPQ} \mathcal{J}^{[I}_M \tilde{\mathcal{J}}^J_N \tilde{\mathcal{J}}^K_P \tilde{\mathcal{J}}^{L]}_Q = 0, \\ \mathcal{E}^{iMN} \mathcal{J}^{[I}_M \tilde{\mathcal{J}}^{J]}_N = 0, \end{cases} \quad (\text{No attractor mechanism}).$$

$$4. \mathcal{E}^{00} = -2\sqrt{2} \langle \mathcal{E}^0 | \Re \left( \mathcal{V}_{IJ} \frac{M^{IJ}}{|M|} \right) \rangle, \text{ (BPS bound)}$$

$$5. \langle \mathcal{E}^0 | \Im \left( \mathcal{V}_{IJ} \frac{M^{IJ}}{|M|} \right) \rangle, \text{ (No NUT charge).}$$

$$6. \begin{cases} \mathcal{E}^{MNPQ} \mathcal{J}^{[I}_M \mathcal{J}^J_N \tilde{\mathcal{J}}^K_P \tilde{\mathcal{J}}^{L]}_Q, \\ \mathcal{E}^{iMN} \mathcal{J}^{[I}_M \mathcal{J}^{J]}_N, \end{cases} \quad \text{are related to } \mathcal{E}^0 \text{ (attractor mechanism).}$$

The precise form of the relation depends on  $N$ :

$$\begin{aligned}
N = 3: \quad \mathcal{E}^{iIJ} &= -2\sqrt{2} \frac{M^{IJ}}{|M|} \langle \mathcal{E}^0 | \mathcal{V}^{*i} \rangle, \\
N = 4: \quad \begin{cases} \mathcal{E}^{IJKL} &= -2\sqrt{2} \frac{M^{[IJ]}}{|M|} \langle \mathcal{E}^0 | \mathcal{V}^{*|KL|} \rangle, \\ \mathcal{E}_{iIJ} &= -2\sqrt{2} \left\{ \frac{M_{IJ}}{|M|} \langle \mathcal{E}^0 | \mathcal{V}_i \rangle + \frac{1}{2} \varepsilon_{IJKL} \frac{M^{KL}}{|M|} \langle \mathcal{E}^0 | \mathcal{V}^{*i} \rangle \right\}, \end{cases} \\
&\text{etc.}
\end{aligned}$$

The only independent equations of motion that have to be imposed on *any*  $d = 4$  supersymmetric configuration are again  $\mathcal{E}^0 = 0$ . Observe that there are scalars which play a rôle analogous to that of the complex scalars in the  $N = 2$  theories and are subject to an attractor mechanism and scalars which play a role analogous to the hyperscalars and are not subject to any such mechanism.

Analogously to the  $N = 2$  case, we find that the construction of any timelike supersymmetric solution proceeds as follows:

**I.** Choose the  $U(2)$  subgroup determining the associated  $N = 2$  *truncation*:

1. Choose  $x$ -dependent rank-2,  $N \times N$  complex antisymmetric  $M_{IJ}$ . With it we construct the projector  $\mathcal{J}^I{}_J \equiv 2|M|^{-2} M^{IK} M_{JK}$ . Supersymmetry requires that

$$\mathfrak{D}\mathcal{J} \equiv d\mathcal{J} - [\mathcal{J}, \Omega] = 0, \quad (3.17)$$

2. Choose three  $N \times N$ , Hermitean, traceless,  $x$ -dependent  $(\sigma^m)^I{}_J$ , satisfying the same properties as the Pauli matrices in the subspace preserved by  $\mathcal{J}$ .

We also have to impose the constraint

$$\mathcal{J} d\sigma^m \mathcal{J} = 0. \quad (3.18)$$

Once the  $U(2)$  subgroup has been chosen, we can split the Vielbeins  $P_{IJKL\mu}$  and  $P_{iIJ\mu}$ , into associated to the would-be vector multiplets in the  $N = 2$  *truncation*

$$P_{IJKL} \mathcal{J}^I{}_{[M} \mathcal{J}^J{}_N \tilde{\mathcal{J}}^K{}_P \tilde{\mathcal{J}}^L{}_{Q]}, \quad \text{and} \quad P_{iIJ} \mathcal{J}^I{}_{[K} \mathcal{J}^J{}_{L]}, \quad (3.19)$$

which are driven by the *attractor mechanism* (i.e. they are determined by the electric and magnetic charges) and those associated to the *hypermultiplets*

$$P_{IJKL} \mathcal{J}^I{}_{[M} \tilde{\mathcal{J}}^J{}_N \tilde{\mathcal{J}}^K{}_P \tilde{\mathcal{J}}^L{}_{Q]}, \quad \text{and} \quad P_{iIJ} \mathcal{J}^I{}_{[K} \tilde{\mathcal{J}}^J{}_{L]}. \quad (3.20)$$

which are not.

In *hyper*-less solutions (e.g. black holes) the  $\sigma^m$ s matrices are not needed at all.

**II.** After the choice of  $U(2)$  subgroup, the solutions are constructed:

1. Define the real symplectic vectors  $\mathcal{R}$  and  $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2} \mathcal{V}_{IJ} M^{IJ}. \quad (3.21)$$

( $U(N)$  singlets  $\Rightarrow$  no  $U(N)$  gauge-fixing necessary)

2. The components of  $\mathcal{I}$  are given by a symplectic vector real functions  $\mathcal{H}$  harmonic in the 3-dimensional transverse space with metric  $\gamma_{mn}$  Eq. (2.11).

3.  $\mathcal{R}$  is to be found from  $\mathcal{I}$  exploiting again the redundancy in the description of the scalars by the sections  $\mathcal{V}_{IJ}, \mathcal{V}_i$ .

4. The metric is

$$ds^2 = |M|^2 (dt + \omega)^2 - |M|^{-2} \gamma_{mn} dx^m dx^n. \quad (3.22)$$

where

$$|M|^{-2} = (M^{IJ} M_{IJ})^{-2} = \langle \mathcal{R} | \mathcal{I} \rangle, \quad (d\omega)_{mn} = 2\epsilon_{mnp} \langle \mathcal{I} | \partial^p \mathcal{I} \rangle. \quad (3.23)$$

$\gamma_{mn}$  is determined indirectly from the would-be *hypers* in the associated  $N = 2$  *truncation* and its curvature vanishes when those scalars vanish.

Its spin connection  $\varpi^{mn}$  is related to  $\Omega$ , by

$$\varpi^{mn} = i\varepsilon^{mnp} \text{Tr} [\sigma^p \Omega]. \quad (3.24)$$

(Observe that only the  $\mathfrak{su}(2)$  components of  $\Omega$  contribute to  $\varpi^{mn}$ .)

5. The vector field strengths are

$$F = -\frac{1}{2} d(\mathcal{R}\hat{V}) - \frac{1}{2} \star (\hat{V} \wedge d\mathcal{I}), \quad \hat{V} = \sqrt{2} |M|^2 (dt + \omega). \quad (3.25)$$

6. The scalars in the *vector multiplets* in the associated  $N = 2$  *truncation*

$$P_{IJKL} \mathcal{J}^I_{[M} \mathcal{J}^J_N \tilde{\mathcal{J}}^K_P \tilde{\mathcal{J}}^L_{Q]}, \quad \text{and} \quad P_{iIJ} \mathcal{J}^I_{[K} \mathcal{J}^J_{L]}, \quad (3.26)$$

can be found from  $\mathcal{R}$  and  $\mathcal{I}$ , while those in the *hypers* must be found independently by solving

$$P_{IJKLm} \mathcal{J}^I_{[M} \tilde{\mathcal{J}}^J_N \tilde{\mathcal{J}}^K_P \tilde{\mathcal{J}}^L_{Q]} (\sigma^m)^Q_R = 0, \quad (3.27)$$

$$P_{iIJm} \mathcal{J}^I_{[K} \tilde{\mathcal{J}}^J_{L]} (\sigma^m)^L_M = 0,$$

which solve their equations of motion according to the KSIs.

## 4 Conclusions

Supersymmetric solutions play an extremely important rôle in many recent developments. We have determined the general form of all the timelike supersymmetric solutions of all ungauged  $d = 4$  supergravities and we have proven the relation between the timelike supersymmetric solutions of all  $d = 4$  supergravities and those of the  $N = 2$  theories (conjectured for black-hole solutions in [15]).

Much work remains to be done in order to make explicit the construction of the solutions: one has to find general parametrizations of the matrices  $M^{IJ}$  and  $\mathcal{J}^I_J$ , solve the *stabilization equations*, impose the covariant constancy of  $\mathcal{J}$  etc. However, this result will allow us to have in explicit, analytic way the most general U-duality invariant families of  $d = 4$  black-hole solutions. Work in this direction is already in progress.

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## References

- [1] P. Meessen and T. Ortín, Phys. Lett. B **693** (2010) 358 [arXiv:1007.3917 [hep-th]].
- [2] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. Lett. **103** (2009) 081301 [arXiv:0905.2326].
- [3] R. Kallosh and T. Ortín, hep-th/9306085. J. Bellorín and T. Ortín, Phys. Lett. B **616** (2005) 118 [hep-th/0501246].
- [4] J. Bellorín, P. Meessen and T. Ortín, Nucl. Phys. B **762** (2007) 229 [hep-th/0606201].
- [5] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D **52** (1995) 5412 [hep-th/9508072]; A. Strominger, Phys. Lett. B **383** (1996) 39 [hep-th/9602111]; S. Ferrara and R. Kallosh, Phys. Rev. D **54** (1996) 1514 [hep-th/9602136], Phys. Rev. D **54** (1996) 1525 [hep-th/9603090].
- [6] K.P. Tod, Phys. Lett. B **121** (1983) 241.
- [7] J. P. Gauntlett and S. Pakis, JHEP **0304** (2003) 039 [hep-th/0212008];
- [8] P. Meessen, T. Ortín and S. Vaulà, arXiv:1006.0239 [hep-th].
- [9] P. Meessen and T. Ortín, Nucl. Phys. B **749** (2006) 291 [hep-th/0603099]. M. Hübscher, P. Meessen and T. Ortín, Nucl. Phys. B **759** (2006) 228 [hep-th/0606281].
- [10] M. M. Caldarelli and D. Klemm, JHEP **0309** (2003) 019 [hep-th/0307022]. S. L. Cacciatori, M. M. Caldarelli, D. Klemm and D. S. Mansi, JHEP **0407** (2004) 061 [hep-th/0406238]. S. L. Cacciatori, D. Klemm, D. S. Mansi and E. Zorzan, JHEP **0805** (2008) 097 [arXiv:0804.0009]. D. Klemm and E. Zorzan, Class. Quant. Grav. **26** (2009) 145018 [arXiv:0902.4186]. D. Klemm and E. Zorzan, arXiv:1003.2974
- [11] M. Hübscher, P. Meessen, T. Ortín and S. Vaulà, Phys. Rev. D **78** (2008) 065031 [arXiv:0712.1530]. P. Meessen, Phys. Lett. B **665** (2008) 388 [arXiv:0803.0684 [hep-th]]. M. Hübscher, P. Meessen, T. Ortín and S. Vaulà, JHEP **0809** (2008) 099 [arXiv:0806.1477].

- [12] R. Bartnik and J. Mckinnon, Phys. Rev. Lett. **61** (1988) 141. P. Bizon, Phys. Rev. Lett. **64** (1990) 2844.
- [13] A. Van Proeyen, “ $N = 2$  supergravity in  $d = 4, 5, 6$  and its matter couplings”, lectures given at the Institute Henri Poincaré, Paris, November 2000. <http://itf.fys.kuleuven.ac.be/~toine/LectParis.pdf>; L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, J. Geom. Phys. **23** (1997) 111 [hep-th/9605032].
- [14] L. Andrianopoli, R. D’Auria and S. Ferrara, Int. J. Mod. Phys. A **13** (1998) 431 [hep-th/9612105].
- [15] S. Ferrara, E. G. Gimon and R. Kallosh, Phys. Rev. D **74** (2006) 125018 [hep-th/0606211].